数理解析研究所講究録
第2028巻 2017年 9-20

MULTIFRACTAL ANALYSIS FOR POINTWISE HÖLDER EXPONENTS OF THE COMPLEX TAKAGI FUNCTIONS IN RANDOM COMPLEX DYNAMICS

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ABSTRACT. We consider hyperbolic random complex dynamical systems on the Riemann sphere with separating condition and multiple minimal sets. We investigate the Hölder regularity of the function T of the probability of tending to one minimal set, the partial derivatives of T with respect to the probability parameters, which can be regarded as complex analogues of the Takagi function, and the higher partial derivatives C of T. Our main result gives a dynamical description of the pointwise Hölder exponents of T and C, which allows us to determine the spectrum of pointwise Hölder exponents by employing the multifractal formalism in ergodic theory. Also, we prove that the bottom of the spectrum α_- is strictly less than 1, which allows us to show that the averaged system acts chaotically on the Banach space C^α of α-Hölder continuous functions for every $\alpha \in (\alpha_-, 1)$, though the averaged system behaves very mildly (e.g. we have spectral gaps) on C^β for small $\beta > 0$.

1. MAIN RESULTS

This note is the summary of the results from paper [JS16]. We do not give any proofs of them in this note. For the proofs of the results, see [JS16]. In this paper, we consider random dynamical systems of rational maps on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$. The study of random complex dynamics was initiated by J.E. Fornaess and N. Sibony ([FS91]). There are many new interesting phenomena in random dynamical systems, so called randomness-induced phenomena or noise-induced phenomena, which cannot hold in the deterministic iteration dynamics. For the motivations and recent research of random complex dynamical systems focused on the randomness-induced phenomena, see the second author's works [Sum11a, Sum13, Sumi15a, Sum15b]. In these papers it was shown that for a generic random dynamical system of complex polynomials, the system acts very mildly on the space of continuous functions on $\widehat{\mathbb{C}}$ and on the space $C^\alpha(\widehat{\mathbb{C}})$ for small $\alpha \in (0,1)$, where $C^\alpha(\widehat{\mathbb{C}})$ denotes the Banach space of α-Hölder continuous functions on $\widehat{\mathbb{C}}$ endowed with α-Hölder norm, but under certain conditions the system still acts chaotically on the space $C^\beta(\widehat{\mathbb{C}})$ for some $\beta \in (0,1)$ close to 1. Thus, we investigate the gradation between chaos and order in random (complex) dynamical systems.

In order to show the main ideas of the paper, let Rat denote the set of all non-constant rational maps on $\widehat{\mathbb{C}}$. This is a semigroup whose semigroup operation is the composition of maps. Throughout the paper, let $s \geq 1$ and let $(f_1, \dots, f_{s+1}) \in (\mathrm{Rat})^{s+1}$ with $\deg(f_i) \geq 2, i = 1, \dots, s+1$. Let $\mathbf{p} = (p_1, \dots, p_s) \in (0,1)^s$ with $\sum_{i=1}^s p_i < 1$ and let $p_{s+1} := 1 - \sum_{i=1}^s p_i$. We consider the (i.i.d.) random dynamical system on $\widehat{\mathbb{C}}$ such that at every step we choose f_i with probability p_i. This defines a Markov chain with state space $\widehat{\mathbb{C}}$ such that for each $x \in \widehat{\mathbb{C}}$ and for each Borel measurable subset A of $\widehat{\mathbb{C}}$, the transition probability $p(x,A)$ from x to A

Date: 8th May 2016. *MSC2010*: 37H10, 37F15.
Key words and phrases. Complex dynamical systems, rational semigroups, random complex dynamics, multifractal formalism, Julia set, random iteration.

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is equal to $\sum_{i=1}^{s+1} p_i 1_A(f_i(x))$, where 1_A denotes the characteristic function of A. Let $G = \langle f_1, \dots, f_s, f_{s+1} \rangle$ be the rational semigroup (i.e., subsemigroup of Rat) generated by $\{f_1, \dots, f_{s+1}\}$. More precisely, $G = \{f_{\omega_n} \circ \cdots \circ f_{\omega_1} : n \in \mathbb{N}, \omega_1, \dots, \omega_n \in \{1, \dots, s+1\}\}$. We denote by $F(G)$ the maximal open subset of $\widehat{\mathbb{C}}$ on which G is equicontinuous with respect to the spherical distance on $\widehat{\mathbb{C}}$. The set $F(G)$ is called the Fatou set of G, and the set $J(G) := \widehat{\mathbb{C}} \setminus F(G)$ is called the Julia set of G. We remark that in order to investigate random complex dynamical systems, it is very important to investigate the dynamics of associated rational semigroups. The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([HM96]), who were interested in the role of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([GR96]), who studied such semigroups from the perspective of random dynamical systems. For the interplay of random complex dynamics and dynamics of rational semigroups, see [Sum00]–[Sum15b], [SS11, SU13, JS15a, JS15b].

Throughout the paper, we assume the following.

(1) G is hyperbolic, i.e., we have $P(G) \subset F(G)$, where

$$P(G) := \overline{\bigcup_{g \in G \cup \{\mathrm{id}\}} g(\cup_{i=1}^{s+1}\{\text{critical values of } f_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}\})}. \text{ Here, the closure is taken in } \widehat{\mathbb{C}}.$$

(2) $(f_1, \dots, f_{s+1})$ satisfies the separating condition, i.e., $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G)) = \varnothing$ whenever $i, j \in \{1, \dots, s+1\}, i \neq j$.

(3) There exist at least two minimal sets of G. Here, a non-empty compact subset K of $\widehat{\mathbb{C}}$ is called a minimal set of G if $K = \overline{\cup_{g \in G}\{g(z)\}}$ for each $z \in K$.

Note that by assumption (2), [Sum97, Lemma 1.1.4] and [Sum11a, Theorem 3.15], we have that there exist at most finitely many minimal sets of G. Moreover, denoting by S_G the union of minimal sets of G and setting $I := \{1, \dots, s+1\}$, we have that for each $z \in \widehat{\mathbb{C}}$ there exists a Borel subset A_z of $I^{\mathbb{N}}$ with $\tilde{\rho}_{\mathbf{p}}(A_z) = 1$ such that $d(f_{\omega_n} \cdots f_{\omega_1}(z), S_G) \to 0$ as $n \to \infty$ for all $\omega = (\omega_i)_{i=1}^{\infty} \in A_z$, where $\tilde{\rho}_{\mathbf{p}} := \otimes_{n=1}^{\infty} \rho_{\mathbf{p}}$ denotes the product measure on $I^{\mathbb{N}}$ given by $\rho_{\mathbf{p}} := \sum_{i=1}^{s+1} p_i \delta_i$ with δ_i denoting the Dirac measure concentrated at $i \in I$.

Throughout, we fix a minimal set L of G (e.g. $L = \{\infty\}$ when G is a polynomial semigroup). Denote by $T_{\mathbf{p}}(z)$ the probability of tending to L of the process on $\widehat{\mathbb{C}}$ which starts in $z \in \widehat{\mathbb{C}}$ and which is given by drawing independently with probability p_i the map f_i. More precisely, $T_{\mathbf{p}}(z) := \tilde{\rho}_{\mathbf{p}}(\{\omega = (\omega_i)_{i=1}^{\infty} \in I^{\mathbb{N}} : d(f_{\omega_n} \circ \cdots \circ f_{\omega_1}(z), L) \to 0 \text{ as } n \to \infty\})$. It was shown by the second author in [Sum13] that, for each $\mathbf{p} = (p_1, \dots, p_s)$ there exists $\alpha \in (0,1)$ such that $\mathbf{x} = (x_1, \dots, x_s) \mapsto T_{(x_1, \dots, x_s, 1 - \sum_{i=1}^{s} x_i)} \in C^{\alpha}(\widehat{\mathbb{C}})$ is real-analytic in a neighbourhood of $\mathbf{p}$, where $C^{\alpha}(\widehat{\mathbb{C}})$ denotes the $\mathbb{C}$-Banach space of α-Hölder continuous $\mathbb{C}$-valued functions on $\widehat{\mathbb{C}}$ endowed with α-Hölder norm $\|\cdot\|_{\alpha}$ (Remark 1.17). Thus it is very natural and important to consider the following. For $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_0^s$ we denote by $C_{\mathbf{n}} \in C^{\alpha}(\widehat{\mathbb{C}})$ the higher order partial derivative of $T_{\mathbf{p}}$ of order $|\mathbf{n}| := \sum_{i=1}^{s} n_i$ with respect to the probability parameters given by

$$C_{\mathbf{n}}(z) := \frac{\partial^{|\mathbf{n}|} T_{(x_1, \dots, x_s, 1 - \sum_{i=1}^{s} x_i)}(z)}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_s^{n_s}}\bigg|_{\mathbf{x}=\mathbf{p}}, \quad z \in \widehat{\mathbb{C}}.$$

These functions are introduced in [Sum13] by the second author. We introduce the $\mathbb{C}$-vector space

$$\mathscr{C} := \operatorname{span}\{C_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}_0^s\} \subset C^{\alpha}(\widehat{\mathbb{C}}),$$

which consists of all the finite complex linear combinations of elements from $\{C_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}_0^s\}$. The first order derivatives are called complex analogues of the Takagi function in [Sum13]. Note that $C_0 = T_{\mathbf{p}}$.

For an element $C \in \mathscr{C}$ and $z \in \widehat{\mathbb{C}}$ the Hölder exponent $\text{Höl}(C, z)$ is given by

$$\text{Höl}(C,z) := \sup\left\{\alpha \in [0,\infty) : \limsup_{y \to z, y \neq z} \frac{|C(y) - C(z)|}{d(y,z)^{\alpha}} < \infty\right\} \in [0,\infty],$$

where d denotes the spherical distance on $\widehat{\mathbb{C}}$. It was shown in [JS15a] that the level sets

$$H(C_0,\alpha):=\{z\in\widehat{\mathbb{C}}:\text{Höl}(C_0,z)=\alpha\},\quad \alpha\in\mathbb{R},$$

satisfy the multifractal formalism. In particular, there exists an interval of parameters (α_-,α_+) such that the Hausdorff dimension of $H(C_0,\alpha)$ is positive and varies real analytically (see Theorem 1.2 below).

The first main result of this paper gives a dynamical description of the pointwise Hölder exponents for an arbitrary $C\in\mathscr{C}$. We say that $C=\sum_{\mathbf{n}\in\mathbb{N}_0^s}\beta_{\mathbf{n}}C_{\mathbf{n}}\in\mathscr{C}$ is non-trivial if there exists $\mathbf{n}\in\mathbb{N}_0^s$ with $\beta_{\mathbf{n}}\neq 0$. It turns out in Theorem 1.1 below that every non-trivial $C\in\mathscr{C}$ has the same pointwise Hölder exponents. To state the result, we define the skew product map (associated with $(f_i)_{i\in I}$) (see [Sum00])

$$\tilde{f}:I^{\mathbb{N}}\times\widehat{\mathbb{C}}\to I^{\mathbb{N}}\times\widehat{\mathbb{C}},\quad \tilde{f}(\omega,z):=(\sigma(\omega),f_{\omega_1}(z)),$$

where $\sigma:I^{\mathbb{N}}\to I^{\mathbb{N}}$ denotes the shift map given by $\sigma(\omega_1,\omega_2,\dots):=(\omega_2,\omega_3,\dots)$, for $\omega=(\omega_1,\omega_2,\dots)\in I^{\mathbb{N}}$. For every $\omega=(\omega_j)_{j\in\mathbb{N}}\in I^{\mathbb{N}}$ and $n\in\mathbb{N}$, let $f_{\omega|_n}:=f_{\omega_n}\circ\cdots\circ f_{\omega_1}$ and we denote by F_ω the maximal open subset of $\widehat{\mathbb{C}}$ on which $\{f_{\omega|_n}\}_{n\in\mathbb{N}}$ is equicontinuous with respect to d. Let $J_\omega:=\widehat{\mathbb{C}}\setminus F_\omega$. The Julia set of $\tilde{f}$ is given by $J(\tilde{f})=\overline{\bigcup_{\omega\in I^{\mathbb{N}}}\{\omega\}\times J_\omega}$ where the closure is taken in $I^{\mathbb{N}}\times\widehat{\mathbb{C}}$. Note that denoting by $\pi:I^{\mathbb{N}}\times\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ the canonical projection, $\pi:J(\tilde{f})\to J(G)$ is a homeomorphism ([Sum11a, Lemma 4.5], [Sum97, Lemma 1.1.4] and assumption (2)) and $\pi\circ\tilde{f}=\sigma\circ\pi$. We introduce the potentials $\tilde{\varphi},\tilde{\psi}:J(\tilde{f})\to\mathbb{R}$ given by

$$\tilde{\varphi}(\omega,z):=-\log\left\|f'_{\omega_1}(z)\right\|,\qquad \tilde{\psi}(\omega,z):=\log p_{\omega_1},$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric on $\widehat{\mathbb{C}}$. Note that $\tilde{f}^{-1}(J(\tilde{f}))=J(\tilde{f})=\tilde{f}(J(\tilde{f}))$ ([Sum00]). We denote by S_n the ergodic sum of the dynamical system $(J(\tilde{f}),\tilde{f})$.

Theorem 1.1. *For every non-trivial $C=\sum_{\mathbf{n}\in\mathbb{N}_0^s}\beta_{\mathbf{n}}C_{\mathbf{n}}\in\mathscr{C}$ we have*

$$\text{Höl}(C,z)=\liminf_{k\to\infty}\frac{S_k\tilde{\psi}(\omega,z)}{S_k\tilde{\varphi}(\omega,z)},\quad \textit{for all }(\omega,z)\in J(\tilde{f}). \tag{1.1}$$

Combining Theorem 1.1 with our results from [JS15a, Theorem 1.2] on the multifractal formalism, we establish the multifractal formalism for the pointwise Hölder exponents of an arbitrary non-trivial $C\in\mathscr{C}$. To state the results, for any non-trivial $C\in\mathscr{C}$ and $\alpha\in\mathbb{R}$ we denote by

$$H(C,\alpha):=\{y\in\widehat{\mathbb{C}}:\text{Höl}(C,y)=\alpha\}$$

the level set of prescribed Hölder exponent α. The range of the multifractal spectrum is given by

$$\alpha_-:=\inf\{\alpha\in\mathbb{R}:H(C,\alpha)\neq\varnothing\}\in\mathbb{R}\quad\text{and}\quad \alpha_+:=\sup\{\alpha\in\mathbb{R}:H(C,\alpha)\neq\varnothing\}\in\mathbb{R}.$$

By Theorem 1.1, the sets $H(C,\alpha)$ coincide for all non-trivial $C\in\mathscr{C}$. Thus, α_- and α_+ do not depend on the choice of a non-trivial $C\in\mathscr{C}$. Also, $\alpha_->0$ ([Sum98, Theorem 2.6], see also Corollary 1.11).

Theorem 1.2. *All of the following hold.*

(1) *Let $C\in\mathscr{C}$ be non-trivial. If $\alpha_-<\alpha_+$ then the Hausdorff dimension function $\alpha\mapsto\dim_H(H(C,\alpha))$, $\alpha\in(\alpha_-,\alpha_+)$, defines a real analytic and strictly concave positive function on (α_-,α_+) with maximum value $\dim_H(J(G))$. If $\alpha_-=\alpha_+$, then we have $H(C,\alpha_-)=J(G)$.*

(2) *We have $\alpha_-=\alpha_+$ if and only if there exist an automorphism $\theta\in\mathrm{Aut}(\widehat{\mathbb{C}})$, complex numbers $(a_i)_{i\in I}$ and $\lambda\in\mathbb{R}$ such that for all $i\in I$ and $z\in\widehat{\mathbb{C}}$,*

$$\theta\circ f_i\circ\theta^{-1}(z)=a_iz^{\pm\deg(f_i)}\quad\textit{and}\quad\log\deg(f_i)=\lambda\log p_i.$$

In the next theorem we determine the actual Hölder class of every non-trivial $C\in\mathscr{C}$.

Theorem 1.3. *For every non-trivial $C \in \mathscr{C}$ and for every $\alpha < \alpha_-$, the function C is α-Hölder continuous on $\widehat{\mathbb{C}}$. Moreover, C_0 is α_--Hölder continuous on $\widehat{\mathbb{C}}$.*

To prove Theorem 1.3 we develop some ideas from [KS08, JKPS09] for interval maps. The relation between the Hölder continuity of singular measures and their multifractal spectra has been first observed in [KS08], where it was shown that the Hölder continuity of the Minkowski's question mark function coincides with the bottom of the Lyapunov spectrum of the Farey map. In [JKPS09] a similar result has been obtained for expanding interval maps.

In the following Theorem 1.4 we prove that $\alpha_- < 1$. This result allows us to give a complete answer to two important problems raised in [Sum13], which greatly improves the previous partial results in [Sum11a, Sum13, JS15a]. The first implication is that, under the assumptions of our paper, every non-trivial $C \in \mathscr{C}$ is not differentiable at every point of a Borel dense subset A of $J(G)$ with $\dim_H(A) > 0$. Secondly, we obtain in Theorem 1.5 that the averaged system still acts chaotically on the space $C^\alpha(\widehat{\mathbb{C}})$ for any $\alpha \in (\alpha_-, 1)$, although the averaged system acts very mildly on the Banach space $C(\widehat{\mathbb{C}})$ of $\mathbb{C}$-valued continuous functions on $\widehat{\mathbb{C}}$ endowed with the supremum norm and on the Banach space $C^\alpha(\widehat{\mathbb{C}})$ for small $\alpha > 0$ (see [Sum97, Lemma 1.1.4], [Sum11a, Theorem 3.15] and [Sum13, Theorem 1.10]). We recall that if $\mathrm{Höl}(C,z) < 1$ then C is not differentiable at z. If $\mathrm{Höl}(C,z) > 1$ then C is differentiable at z and the derivative of C at z is zero.

Theorem 1.4. *We have $\alpha_- < 1$. Moreover, for every $\alpha \in (\alpha_-, 1)$ there exists a Borel dense subset A of $J(G)$ with $\dim_H(A) > 0$ such that for every non-trivial $C \in \mathscr{C}$ and for every $z \in A$, we have $\mathrm{Höl}(C,z) = \alpha < 1$ and C is not differentiable at z.*

In the proof, we combine the result that C_0 is α_--Hölder continuous on $\widehat{\mathbb{C}}$ (Theorem 1.3), the multifractal analysis on the pointwise Hölder exponents of C_0 (Theorems 1.2), an argument on Lipschitz functions on $\mathbb{C}$ and the fact that $\dim_H(J(G)) < 2$, which follows from our assumptions (1) and (2) ([Sum98]).

To state Theorem 1.5, let $M : C(\widehat{\mathbb{C}}) \to C(\widehat{\mathbb{C}})$ be the transition operator of the system which is defined by $M(\phi)(z) = \sum_{j=1}^{s+1} p_j \phi(f_j(z))$, where $\phi \in C(\widehat{\mathbb{C}}), z \in \widehat{\mathbb{C}}$. Note that $M(C^\alpha(\widehat{\mathbb{C}})) \subset C^\alpha(\widehat{\mathbb{C}})$ for any $\alpha \in (0,1]$.

Theorem 1.5. *Let $\alpha \in (\alpha_-, 1)$ and let $\phi \in C^\alpha(\widehat{\mathbb{C}})$ such that $\phi|_L = 1$ and $\phi|_{L'} = 0$ for every minimal set L' of G with $L' \neq L$. Then $\|M^n(\phi)\|_\alpha \to \infty$ as $n \to \infty$. In particular, for every $\xi \in C^\alpha(\widehat{\mathbb{C}})$ and for every $a \in \mathbb{C} \setminus \{0\}$, we have $\|M^n(\xi + a\phi) - M^n(\xi)\|_\alpha \to \infty$ as $n \to \infty$.*

We now present the corollaries of our main results. The first one establishes that every non-trivial $C \in \mathscr{C}$ varies precisely on the Julia set $J(G)$. This follows immediately from Theorem 1.1 because the right-hand side of (1.1) is always finite ([Sum98, Theorem 2.6], see also Corollary 1.11). This generalises a previous result from [Sum11a] for $C_0 = T_{\mathbf{p}}$ and a partial result for the higher order partial derivatives from [Sum13].

Corollary 1.6. *Every non-trivial $C \in \mathscr{C}$ varies precisely on $J(G)$, i.e., $J(G)$ is equal to the set of points $z_0 \in \widehat{\mathbb{C}}$ such that C is not constant in any neighborhood of z_0 in $\widehat{\mathbb{C}}$. In particular, the functions $C_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}_0^s$, are linearly independent over $\mathbb{C}$ and $\mathscr{C}$ has a representation as a direct sum of vector spaces given by*

$$\mathscr{C} = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^s} \mathbb{C} C_{\mathbf{n}}.$$

We remark again that $0 < \dim_H(J(G)) < 2$ ([Sum98]).

By combining Theorem 1.1 with Birkhoff's ergodic theorem we obtain the following extension of [Sum13, Theorem 3.40 (2)]. Recall that a Borel probability measure ν on $J(\tilde{f})$ is called $\tilde{f}$-invariant if $\nu(\tilde{f}^{-1}(A)) = \nu(A)$ for every Borel set $A \in J(\tilde{f})$.

Corollary 1.7. *Let ν be an $\tilde{f}$-invariant ergodic Borel probability measure on $J(\tilde{f})$. Let $\pi : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the canonical projection onto $\widehat{\mathbb{C}}$. Then there exists a Borel subset A of $J(G)$ with $(\pi_*(\nu))(A) = 1$ such that for every non-trivial $C \in \mathscr{C}$ and for every $z \in A$, we have*

$$\text{Höl}(C,z) = \frac{-\int \log p_{\omega_1} d\nu(\omega,x)}{\int \log \|f'_{\omega_1}(x)\| d\nu(\omega,x)}, \text{ where } \omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}.$$

By combining Corollary 1.7 with [Sum11a, Theorem 3.82] in which the potential theory was used, we obtain the following result (Corollary 1.8) on the pointwise Hölder exponents and the non-differentiability of elements of $\mathscr{C}$. To state the result, when G is a polynomial semigroup, we denote by $\tilde{\mu}_{\mathbf{p}}$ the maximal relative entropy measure on $J(\tilde{f})$ for $\tilde{f}$ with respect to $(\sigma, \tilde{\rho}_{\mathbf{p}})$ (see [Sum00], [Sum11a, Remark 3.79]). Note that $\tilde{\mu}_{\mathbf{p}}$ is $\tilde{f}$-invariant and ergodic ([Sum00]). Let $\mu_{\mathbf{p}} = \pi_*(\tilde{\mu}_{\mathbf{p}})$. For any $(\omega, z) \in I^{\mathbb{N}} \times \widehat{\mathbb{C}}$, let $\mathscr{G}_\omega(z) := \lim_{n\to\infty}(1/\deg(f_{\omega|_n}))\log^+|f_{\omega|_n}(z)|$, where $\log^+(a) := \max\{\log a, 0\}$ for every $a > 0$. By the argument in [Ses01], we have that $\mathscr{G}_\omega(y)$ exists for every $(\omega, z) \in I^{\mathbb{N}} \times \mathbb{C}$, $(\omega, z) \in I^{\mathbb{N}} \times \mathbb{C} \mapsto \mathscr{G}_\omega(z)$ is continuous on $I^{\mathbb{N}} \times \mathbb{C}$, $\mathscr{G}_\omega$ is subharmonic on $\mathbb{C}$ and $\mathscr{G}_\omega$ restricted to the intersection of $\mathbb{C}$ and the basin $A_{\infty,\omega}$ of ∞ for $\{f_{\omega|_n}\}_{n=1}^\infty$ is the Green's function on $A_{\infty,\omega}$ with pole at ∞. Let $\Lambda(\omega) = \sum_c \mathscr{G}_\omega(c)$, where c runs over all critical points of f_{ω_1} in $A_{\infty,\omega}$, counting multiplicities. Note that $\mu_{\mathbf{p}} = \int_{I^{\mathbb{N}}} dd^c \mathscr{G}_\omega d\tilde{\rho}_{\mathbf{p}}(\omega)$ where $d^c = (\sqrt{-1}/2\pi)(\overline{\partial} - \partial)$ ([Sum11a, Lemma 5.51]), $\operatorname{supp}\mu_{\mathbf{p}} = J(G)$ and $\mu_{\mathbf{p}}$ is non-atomic ([Sum00]). Also, we have $\dim_H(\mu_{\mathbf{p}}) = (\sum_{i\in I} p_i \log\deg f_i - \sum_{i\in I} p_i \log p_i)/(\sum_{i\in I} p_i \log\deg f_i + \int_{I^{\mathbb{N}}} \Lambda(\omega) d\tilde{\rho}_{\mathbf{p}}(\omega)) > 0$ ([Sum11a, Proof of Theorem 3.82]). Here, $\dim_H(\mu_{\mathbf{p}}) := \inf\{\dim_H(A)\}$ where the infimum is taken over all Borel subsets A of $J(G)$ with $\mu_{\mathbf{p}}(A) = 1$.

Corollary 1.8. (1) *Suppose that $f_1, \dots, f_{s+1}$ are polynomials. Then there exists a Borel dense subset A of $J(G)$ with $\mu_{\mathbf{p}}(A) = 1$ and $\dim_H(A) \geq (\sum_{i\in I} p_i \log\deg f_i - \sum_{i\in I} p_i \log p_i)/(\sum_{i\in I} p_i \log\deg f_i + \int_{I^{\mathbb{N}}} \Lambda(\omega) d\tilde{\rho}_{\mathbf{p}}(\omega))$ > 0 such that for every non-trivial $C \in \mathscr{C}$ and for every $z \in A$, we have*

$$\text{Höl}(C,z) = \frac{-\sum_{i\in I} p_i \log p_i}{\sum_{i\in I} p_i \log\deg f_i + \int_{I^{\mathbb{N}}} \Lambda(\omega) d\tilde{\rho}_{\mathbf{p}}(\omega)}.$$

(2) *Suppose that $f_1, \dots, f_{s+1}$ are polynomials satisfying at least one of the following conditions:*

- *(a)* $\sum_{i\in I} p_i \log(p_i \log f_i) > 0$.
- *(b)* $G = \langle f_1, \dots, f_{s+1} \rangle$ *is postcritically bounded, i.e.* $P(G) \setminus \{\infty\}$ *is bounded in* $\mathbb{C}$.
- *(c)* $s = 1$.

Then there exists a Borel dense subset A of $J(G)$ with $\mu_{\mathbf{p}}(A) = 1$ such that for every non-trivial $C \in \mathscr{C}$ and for every $z \in A$, we have $\text{Höl}(C,z) < 1$. *In particular, every non-trivial $C \in \mathscr{C}$ is non-differentiable $\mu_{\mathbf{p}}$-almost everywhere on $J(G)$.*

Note that if we assume that every f_i is a polynomial and $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, then $\Lambda(\omega) = 0$ for every $\omega \in I^{\mathbb{N}}$, thus Corollary 1.8 implies that there exists a Borel dense subset A of $J(G)$ with

$$\mu_{\mathbf{p}}(A) = 1, \ \dim_H(A) \geq 1 + \frac{-\sum_{i\in I} p_i \log p_i}{\sum_{i\in I} p_i \log\deg(f_i)} > 1$$

such that for every non-trivial $C \in \mathscr{C}$ and for every point $z \in A$, we have

$$\text{Höl}(C,z) = \frac{-\sum_{i\in I} p_i \log p_i}{\sum_{i\in I} p_i \log\deg(f_i)} < 1.$$

The following is one of the other important applications of Corollary 1.7. In order to state the result, let $\delta := \dim_H(J(G))$ and let H^δ denote the δ-dimensional Hausdorff measure on $\widehat{\mathbb{C}}$. Note that by [Sum05], we have $0 < H^\delta(J(G)) < \infty$. Let $C(J(G))$ be the space of all continuous $\mathbb{C}$-valued functions on $\widehat{\mathbb{C}}$ endowed with supremum norm. Let $L : C(J(G)) \to C(J(G))$ be the operator defined by $L(\varphi)(z) =$

$\sum_{i\in I}\sum_{f_i(y)=z}\phi(y)\|f_i'(y)\|^{-\delta}$ where $\phi\in C(J(G)), z\in J(G)$. By [Sum05] again, we have that $\gamma=\lim_{n\to\infty}L^n(1)\in C(J(G))$ exists, where 1 denotes the constant function on $J(G)$ taking its value 1, the function γ is positive on $J(G)$, and there exists an $\tilde{f}$-invariant ergodic probability measure $\tilde{\nu}$ on $J(\tilde{f})$ such that $\pi_*(\tilde{\nu})=\gamma H^\delta/H^\delta(J(G))$ and $\operatorname{supp}\pi_*(\nu)=J(G)$. By Corollary 1.7 and [Sum11a, Theorem 3.84 (5)], we obtain the following.

Corollary 1.9. *Under the above notations, there exists a Borel dense subset A of $J(G)$ with $H^\delta(A)=H^\delta(J(G))>0$ such that for every non-trivial $C\in\mathscr{C}$ and for every $z\in A$, we have*

$$\text{Höl}(C,z)=\frac{-\sum_{i\in I}\log p_i\int_{f_i^{-1}(J(G))}\gamma(y)dH^\delta(y)}{\sum_{i\in I}\int_{f_i^{-1}(J(G))}\gamma(y)\log\|f_i'(y)\|dH^\delta(y)}.$$

Remark 1.10. We remark that a non-trivial $C\in\mathscr{C}$ may possess points of differentiability. In fact, by choosing one of the probability parameters sufficiently small, we can deduce from Corollary 1.9 that for every non-trivial $C\in\mathscr{C}$ and for H^δ-almost every $z\in J(G)$, we have $\text{Höl}(C,z)>1$, C is differentiable at z and the derivative of C at z is zero. Note that even under the above condition, Theorem 1.4 implies that there exist an $\alpha<1$ and a dense subset A of $J(G)$ with $\dim_H(A)>0$ such that for every non-trivial $C\in\mathbb{C}$ and for every $z\in A$, we have $\text{Höl}(C,z)=\alpha<1$ and C is not differentiable at z. In particular, in this case, we have $\alpha_-<1<\alpha_+$ and we have a different kind of phenomenon regarding the (complex) analogues of the Takagi function, whereas the original Takagi function does not have this property.

We also have the following corollary of Theorem 1.1. To state the result, by [Sum98, Theorem 2.6] there exists $k_0\in\mathbb{N}$ such that for every $k\geq k_0$ and for every $\omega=(\omega_i)_{i=1}^k\in I^k$, we have $\min_{z\in f_\omega^{-1}(J(G))}\|f_\omega'(z)\|>1$, where $f_\omega=f_{\omega_k}\circ\cdots\circ f_{\omega_1}$. Let $p_\omega:=p_{\omega_k}\cdots p_{\omega_1}$ for $\omega=(\omega_i)_{i=1}^k\in I^k$.

Corollary 1.11. *For every $k\geq k_0$, we have*

$$0<\min_{\omega\in I^k}\frac{-\log p_\omega}{\log\max_{z\in f_\omega^{-1}(J(G))}\|f_\omega'(z)\|}\leq\alpha_-\leq\alpha_+\leq\max_{\omega\in I^k}\frac{-\log p_\omega}{\log\min_{z\in f_\omega^{-1}(J(G))}\|f_\omega'(z)\|}<\infty.$$

In particular, if $p_i\min_{z\in f_i^{-1}(J(G))}\|f_i'(z)\|>1$ for every $i\in I$, then for every non-trivial $C\in\mathscr{C}$ and for every $z\in J(G)$, we have that $\text{Höl}(C,z)\leq\alpha_+<1$ *and C is not differentiable at z.*

Remark 1.12. Under assumptions (1)(2)(3), suppose that the maps $f_i, i\in I$, are polynomials. Then $J(G)\subset\mathbb{C}$. Since the spherical metric and the Euclidian metric are equivalent on $J(G)$, it follows that we can replace $\|\cdot\|$ in the definition of φ, Corollaries 1.7, 1.9, 1.11 by the modulus $|\cdot|$.

Remark 1.13. The function $C_0=T_\mathbf{p}$ is continuous (in fact, it is Hölder continuous) on $\widehat{\mathbb{C}}$ and varies precisely on the Julia set $J(G)$. Note that by assumptions (1)(2) and [Sum98], we have that $J(G)$ is a fractal set with $0<\dim_H(J(G))<2$. The function C_0 can be interpreted as a complex analogue of the devil's staircase and Lebesgue's singular functions ([Sum11a]). In fact, the devil's staircase is equal to the restriction to $[0,1]$ of the function of probability of tending to $+\infty$ when we consider random dynamical system on $\mathbb{R}$ such that at every step we choose $f_1(x)=3x$ with probability $1/2$ and we choose $f_2(x)=3x-2$ with probability $1/2$. Similarly, Lebesgue's singular function L_p with respect to the parameter $p\in(0,1), p\neq 1/2$ is equal to the restriction to $[0,1]$ of the function of probability of tending to $+\infty$ when we consider random dynamical system on $\mathbb{R}$ such that at every step we choose $g_1(x)=2x$ with probability p and we choose $g_2(x)=2x-1$ with probability $1-p$. Note that these are new interpretations of the devil's staircase and Lebesgue's singular functions obtained in [Sum11a] by the second author of this paper. Similarly, it was pointed out by him that the distributional functions of self-similar measures of IFSs of orientation-preserving contracting diffeomorphisms h_i on $\mathbb{R}$ can be interpreted as the functions of probability of tending to $+\infty$ regarding the random dynamical systems generated by (h_i^{-1}) ([Sum11a]). From the above point of view, when G is a polynomial semigroup and $L=\{\infty\}$, we call $C_0 = T_{\mathbf{p}}$ a devil's coliseum ([Sum11a]). It is well-known ([YHK97]) that the function $\frac{1}{2}\frac{\partial L_p(x)}{\partial p}|_{p=1/2}$ on $[0,1]$ is equal to the Takagi function $\Phi(x) = \sum_{n=0}^{\infty}\frac{1}{2^n}\min_{m\in\mathbb{Z}}|2^n x - m|$ (also referred to as the Blancmange function), which is a famous example of a continuous but nowhere differentiable function on $[0,1]$. From this point of view, the first derivatives $C \in \mathscr{C}$ can be interpreted as complex analogues of the Takagi function. The devil's staircase, Lebesgue's singular functions, the Takagi function and the similar functions have been investigated so long in fractal geometry and the related fields. In fact, the graphs of these functions have certain kind of self-similarities and these functions have many interesting and deep properties. There are many interesting studies about the original Takagi function and its related topics ([AK11]). In [AK06], many interesting results (e.g. continuity and non-differentiability, Hölder order, the Hausdorff dimension of the graph, the set of points where the functions take on their absolute maximum and minimum values) of the higher order partial derivatives $\frac{\partial^n L_p(x)}{\partial p^n}|_{p=1/2}$ of $L_p(x)$ with respect to p are obtained. The first study of the complex analogues of the Takagi function was given by the second author in [Sum13]. In particular, some partial results on the pointwise Hölder exponents of them were obtained ([Sum13, Theorem 3.40]). However, it had been an open problem whether the complex analogues of the Takagi function vary precisely on the Julia set or not, until this paper was written. The results of this paper greatly improve the above results from [Sum13]. In the proofs of the results of this paper, we use completely new ideas and systematic approaches which are explained below. For the figures of the Julia set $J(G)$ and the graphs of C_0 and C_1 which we deal with in this paper when $s=1$, G is a polynomial semigroup and $L=\{\infty\}$, see [Sum11a, Sum13].

Remark 1.14. The results on the classical Takagi function on $[0,1]$ give some evidence that the results stated in Theorem 1.3 are sharp. Indeed, let us consider the function $L_{1/2}$ and $\phi_n(x) = \frac{\partial^n L_p(x)}{\partial p^n}|_{p=1/2}$ for $n \geq 1$. Note that $\frac{1}{2}\phi_1$ is equal to the original Takagi function. Since we have $L_{1/2}|_{[0,1]}(x) = x$, $L_{1/2}|_{(-\infty,0)}(x) = 0$ and $L_{1/2}|_{(1,\infty)}(x) = 1$, the function $L_{1/2}$ is 1-Hölder (Lipschitz). However, in [AK06] it is shown that the functions ϕ_n on $[0,1]$ are a-Hölder for every $a<1$, but not 1-Hölder continuous. It would be interesting to further investigate this phenomenon for the complex analogues of the Takagi function.

Remark 1.15. We endow Rat with the topology induced from the distance $\mathrm{dist}_{\mathrm{Rat}}$ which is defined by $\mathrm{dist}_{\mathrm{Rat}}(f,g) := \sup_{z\in\widehat{\mathbb{C}}} d(f(z),g(z))$. Then by [Sum97, Theorem 2.4.1], the fact $J(G) = \cup_{i\in I} f_i^{-1}(J(G))$ ([Sum97, Lemma 1.1.4]), [Sum11a, Remark 3.64], and [Sum13, Theorem 3.24]), we have that the set

$$\{(f_i)_{i\in I} \in (\mathrm{Rat})^I : \deg(f_i) \geq 2\ (i \in I) \text{ and the conditions (1)(2)(3) hold for } (f_i)_{i\in I}\}$$

is open in $(\mathrm{Rat})^I$. Also, we have plenty of examples to which we can apply the main results of this paper. See Section 2.

Remark 1.16. We remark that by using the method in this paper, we can show similar results to those of this paper for random dynamical systems of diffeomorphisms on $\mathbb{R}$ (or $\mathbb{R}\cup\{\pm\infty\}$). Note that the case of the classical Takagi function Φ corresponds to the degenerated case $\alpha_- = \alpha_+$ in Theorem 1.2, though in the case of Φ we have the open set condition but do not have the separating condition. We emphasize that in this paper we also deal with the non-degenerated case, which seems generic.

Remark 1.17. We remark that under assumptions (1)(2)(3), the iteration of the transition operator M on some $C^a(\widehat{\mathbb{C}})$ is well-behaved (e.g., there exists an M-invariant finite-dimensional subspace U of $C^a(\widehat{\mathbb{C}})$ such that for every $h \in C^a(\widehat{\mathbb{C}})$, $M^n(h)$ tends to U as $n \to \infty$ exponentially fast) and M has a spectral gap on $C^a(\widehat{\mathbb{C}})$ ([Sum97, Lemma 1.1.4(2)], [Sum11a, Propositions 3.63, 3.65], [Sum13, Theorems 3.30, 3.31]). Note that this is a randomness-induced phenomenon (new phenomenon) in random dynamical systems which cannot hold in the deterministic iteration dynamics of rational maps of degree two or more, since for every $f \in \text{Rat}$ with $\deg(f) \geq 2$, the dynamics of f on $J(f)$ is chaotic. Combining the above spectral gap property of M on $C^a(\widehat{\mathbb{C}})$ and the perturbation theory for linear operators ([Kato80]) implies that the map $\mathbf{x} = (x_1,\dots,x_s) \mapsto T_{(x_1,\dots,x_s,1-\sum_{i=1}^s x_i)} \in C^a(\widehat{\mathbb{C}})$ is real-analytic in a neighbourhood of $\mathbf{p}$ in the space $W := \{(q_i)_{i=1}^s \in (0,1)^s : \sum_{i=1}^s q_i < 1\}$ ([Sum13, Theorem 3.32]). Thus it is very natural and important for the study of the random dynamical system to consider the higher order partial derivatives of $T_\mathbf{p}$ with respect to the probability vectors. Moreover, it is very interesting that $C_\mathbf{n}$ is a solution of the functional equation $(Id - M)(C_\mathbf{n}) = F$, where F is a function associated with lower order partial derivatives of $T_\mathbf{p}$. In fact, by using the spectral gap properties of M on $C^a(\widehat{\mathbb{C}})$ and the arguments in the proof of [Sum13, Theorem 3.32], for any $\mathbf{n} \in \mathbb{N}_0^s \setminus \{0\}$, we can show that (I) $C_\mathbf{n}$ is the unique continuous solution of the above functional equation under the boundary condition $C_\mathbf{n}|_{S_G} = 0$ and (II) $C_\mathbf{n} = \sum_{j=0}^{\infty} M^j(F)$ in $C(\widehat{\mathbb{C}})$ and in $C^\alpha(\widehat{\mathbb{C}})$ for small $\alpha > 0$. Thus, we have a system of functional equations for elements $C_\mathbf{n}$. Note that this is the first paper to investigate the pointwise Hölder exponents and other properties of the higher order partial derivatives $C_\mathbf{n}$ of the functions $T_\mathbf{p}$ of probability of tending to minimal sets with respect to the probability parameters regarding random dynamical systems which have several variables of probability parameters. This is a completely new concept. In fact, even in the real line, there has been no study regarding the objects similar to the above. Even more, in this paper we deal with the complex linear combinations of partial derivatives $C_\mathbf{n}$, which are of course completely new objects in mathematics coming naturally from the study of random dynamical systems and fractal geometry. We also remark that the original Takagi function is associated with Lebesgue's singular functions, but there has been no study about the higher order partial derivatives of the distribution functions of singular measures with respect to the parameters.

The key in the proof of the main results of this paper is to consider the system of functional equations satisfied by the elements of $\mathscr{C}$. The composition of these equations along orbits is best described in terms of an associated matrix cocycle $A(\omega,k)$. By using combinatorial arguments, we show a formula for the components of the matrix $A(\omega,k)$, and we carefully estimate the polynomial growth order of these components, as k tends to infinity. Combining this with some calculations of the determinants of matrices which are similar to the Vandermonde determinant, we deduce the linear independence of the vectors $(C_\mathbf{r}(a) - C_\mathbf{r}(b))_{\mathbf{r}\leq\mathbf{n}}$ for certain points $a,b \in J(G)$ which are close to a given point $x_0 \in J(G)$. Here, $\mathbf{r} \leq \mathbf{n}$ means that $r_i \leq n_i$ for each i. From the linear independence of these vectors we deduce that a certain linear combination of vectors $(C_\mathbf{r}(a) - C_\mathbf{r}(b))_{\mathbf{r}\leq\mathbf{n}}$ is bounded away from zero. This gives us the upper bound of the pointwise Hölder exponents of $C \in \mathscr{C}$. Note that this argument is the key to prove Theorem 1.1 and it is the crucial point to derive that the elements $C \in \mathscr{C}$ are not locally constant in any point of the Julia set (Corollary 1.6). We emphasize that those ideas are very new and they give us strong and systematic tools to analyze random dynamical systems, singular functions, fractal functions and other related topics.

2. EXAMPLES

In this section, we give some examples which illustrate the main results of this paper.

For $f \in$Rat, we set $F(f) := F(\langle f\rangle), J(f) := J(\langle f\rangle)$, and $P(f) = P(\langle f\rangle)$. We denote by $\mathscr{P}$ the set of polynomials of degree two or more. For $g \in \mathscr{P}$, we denote by $K(g)$ the filled-in Julia set. If G is a rational semigroup and if K is a non-empty compact subset of $\widehat{\mathbb{C}}$ such that $g(K) \subset K$ for each $g \in G$, then Zorn's lemma implies that there exists a minimal set L of G with $L \subset K$ ([Sum11a, Remark 3.9]).

The following propositions show us several methods to produce many examples of $(f_1,\dots,f_{s+1}) \in (\text{Rat})^{s+1}$ which satisfy assumptions (1)(2)(3) of this paper. For such elements $(f_1,\dots,f_{s+1})$ and for every $\mathbf{p} =$

$(p_i)_{i=1}^s \in (0,1)^s$ with $\sum_{i=1}^s p_i < 1$, we can apply the results Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and Corollaries 1.6, 1.7, 1.9 and 1.11 in Section 1.

Proposition 2.1. *Let $(g_1,\dots,g_{s+1}) \in (\mathrm{Rat})^{s+1}$ with $\deg(g_i) \geq 2, i = 1\dots,s+1$. Suppose that $\langle g_1,\dots,g_{s+1}\rangle$ is hyperbolic, $J(g_i)\cap J(g_j) = \varnothing$ for every (i,j) with $i \neq j$, and that there exist at least two distinct minimal sets of $\langle g_1,\dots,g_{s+1}\rangle$. Then there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq m$, setting $f_i = g_i^n, i = 1,\dots,s+1$, the element $(f_1,\dots,f_{s+1})$ satisfies assumptions (1)(2)(3) of this paper.*

Proposition 2.2. *Let $(g_1,\dots,g_{s+1}) \in (\mathrm{Rat})^{s+1}$ with $\deg(g_i) \geq 2, i = 1,\dots,s+1$. Suppose that $\cup_{i=1}^{s+1}P(g_i) \subset \cap_{i=1}^{s+1}F(g_i)$, that $J(g_i)\cap J(g_j) = \varnothing$ for every (i,j) with $i \neq j$, and that there exist two compact subsets K_1, K_2 of $\widehat{\mathbb{C}}$ with $K_1 \cap K_2 = \varnothing$ such that $g_i(K_j) \subset K_j$ for every $i = 1,\dots,s+1$ and for $j = 1,2$. Then there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq m$, setting $f_i = g_i^n, i = 1,\dots,s+1$, the element $(f_1,\dots,f_{s+1})$ satisfies assumptions (1)(2)(3) of this paper.*

Combining [Sum11a, Remark 3.9] with [Sum11a, Proposition 6.1], we also obtain the following.

Proposition 2.3. *Let $f_1 \in \mathscr{P}$ be hyperbolic, i.e., $P(f_1) \subset F(f_1)$. Suppose that $\mathrm{Int}(K(f_1)) \neq \varnothing$, where* Int *denotes the set of interior points. Let $b \in \mathrm{Int}(K(f_1))$ be a point. Let $d \in \mathbb{N}$ with $d \geq 2$. Suppose that $(\deg(f_1),d) \neq (2,2)$. Then there exists a number $c > 0$ such that for each $\lambda \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < c\}$, setting $f_{2,\lambda}(z) := \lambda(z-b)^d + b$, we have the following.*

(1) *$(f_1, f_{2,\lambda})$ satisfies assumptions* (1)(2)(3) *of this paper with $s = 1$.*
(2) *If $J(f_1)$ is connected, then $P(\langle f_1, f_{2,\lambda}\rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$.*

Thus combining the above with Remark 1.15, we obtain that for any $(f_1, f_{2,\lambda})$ in the above, there exists a neighborhood V of $(f_1, f_{2,\lambda})$ in $(\mathrm{Rat})^2$ such that for every $(g_1,g_2) \in V$, assumptions (1)(2)(3) of this paper are satisfied and Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and Corollaries 1.6, 1.7,1.9 and 1.11 in Section 1 hold. Also, endowing $\mathscr{P}$ with the relative topology from Rat, we have that there exists an open neighborhood W of $(f_1, f_{2,\lambda})$ in $\mathscr{P}^2$ such that for every $(g_1,g_2) \in W$ and for every $\mathbf{p} = p_1 \in (0,1)$, Corollary 1.8 holds.

Example 2.4. Let $(f_1,f_2) \in \mathscr{P}^2$ be an element such that $\langle f_1,f_2\rangle$ is hyperbolic, $P(\langle f_1,f_2\rangle)\setminus\{\infty\}$ is bounded in $\mathbb{C}$ and $J(\langle f_1,f_2\rangle)$ is disconnected. Note that there are plenty of examples of such elements (f_1,f_2) (Proposition 2.3, [Sum11b, Sum15b]). Then by [Sum09, Theorems 1.5, 1.7], we have that $f_1^{-1}(J(G)) \cap f_2^{-1}(J(G)) = \varnothing$ where $G = \langle f_1,f_2\rangle$. Thus (f_1,f_2) satisfies assumptions (1)(2)(3) of this paper with $s = 1$ and all results in Section 1 hold for (f_1,f_2) and for every $\mathbf{p} = p_1 \in (0,1)$.

Example 2.5. Let $g_1(z) = z^2-1, g_2(z) = z^2/4$, and let $f_i = g_i \circ g_i$, $i = 1,2$. Let $\mathbf{p} = p_1 = 1/2$. Let $G = \langle f_1,f_2\rangle$. Then (f_1,f_2) satisfies the assumptions (1)(2)(3) of this paper with $s = 1$ and $P(G)\setminus\{\infty\}$ is bounded in $\mathbb{C}$ ([Sum11a, Example 6.2],[Sum13, Example 6.2]). Thus for this (f_1,f_2), all results of Section 1 hold. In particular, every non-trivial $C \in \mathscr{C}$ is Hölder continuous on $\widehat{\mathbb{C}}$ and varies precisely on the Julia set $J(G)$ (Corollary 1.6). Moreover, by Corollary 1.8, there exists a Borel dense subset A of $J(G)$ with $\mu_{\mathbf{p}}(A) = 1$, $\dim_H(A) \geq \dim_H(\mu_{\mathbf{p}}) = \frac{3}{2}$ such that for every non-trivial $C \in \mathscr{C}$ and for every $z \in A$, we have $\alpha_- \leq \mathrm{H\ddot{o}l}(C,z) = \frac{1}{2} \leq \alpha_+$ and C is not differentiable at z. For the figures of $J(G)$ and the graphs of C_0, C_1 with $L = \{\infty\}$, see [Sum13, Figures 2,3,4]. Note that Theorem 1.2 implies that $\alpha_- < \alpha_+$ for every probability vector (parameter) $\mathbf{p}' \in (0,1)$.

Example 2.6. Let $\lambda \in \mathbb{C}$ with $0 < |\lambda| \leq 0.01$ and let $f_1(z) = z^2-1, f_2(z) = \lambda z^3$. Then by [Sumi15a, Example 5.4], the element (f_1,f_2) satisfies assumptions (1)(2)(3) of this paper with $s = 1$ and $P(\langle f_1,f_2\rangle)\setminus\{\infty\}$ is bounded in $\mathbb{C}$. Thus all results in Section 1 hold for (f_1,f_2) and for every probability vector (parameter)

$\mathbf{p} = p_1 \in (0,1)$. Thus, setting $p_1 = \frac{1}{2}$, $G = \langle f_1, f_2 \rangle$ and $L = \{\infty\}$, every non-trivial $C \in \mathscr{C}$ is Hölder continuous on $\hat{\mathbb{C}}$ and varies precisely on $J(G)$, and Corollary 1.8 implies that there exists a Borel dense subset A of $J(G)$ with $\mu_{\mathbf{p}}(A) = 1$ and $\dim_H(A) \geq 1 + \frac{2\log 2}{\log 2 + \log 3} \fallingdotseq 1.7737$ such that for every non-trivial $C \in \mathscr{C}$ and for every $z \in A$, we have $\alpha_- \leq \mathrm{Höl}(C,z) = \frac{2\log 2}{\log 2+\log 3} (\fallingdotseq 0.7737) \leq \alpha_+$ and C is not differentiable at z. Also, by Theorem 1.2, we have $\alpha_- < \alpha_+$ for every $\mathbf{p}' \in (0,1)$.

Example 2.7. Let $g_1, g_2 \in \mathscr{P}$ be hyperbolic. Suppose that $(J(g_1) \cup J(g_2)) \cap (P(g_1) \cup P(g_2)) = \emptyset$, $K(g_1) \subset \mathrm{Int}(K(g_2))$, and the union of attracting cycles of g_2 in $\mathbb{C}$ is included in $\mathrm{Int}(K(g_1))$. Then by [Sum11a, Proposition 6.3], there exists an $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq m$, setting $f_1 = g_1^n, f_2 = g_2^n$, we have that (f_1, f_2) satisfies assumptions (1)(2)(3) of this paper with $s = 1$. Thus all statements of the results in Section 1 hold for (f_1, f_2) and for every $\mathbf{p} = p_1 \in (0,1)$.

The following proposition provides us a method to construct examples of $(f_1, \dots, f_{s+1}) \in \mathscr{P}^{s+1}$ for which (1)(2)(3) hold and $P(\langle f_1, \dots, f_{s+1} \rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. For such elements $(f_1, \dots, f_{s+1})$ and for every $\mathbf{p} \in (0,1)^s$ with $\sum_{i=1}^s p_i < 1$, we can apply all the results in Section 1.

Proposition 2.8. *Let $g_1, \dots, g_{s+1} \in \mathscr{P}$ be hyperbolic and suppose that $J(f_i)$ is connected for every $i = 1, \dots, s+1$. Suppose that $J(f_i) \subset \mathrm{Int}(K(f_{i+1}))$ for every $i = 1, \dots, s$. Suppose also that $\cup_{i=2}^{s+1} P(g_i) \setminus \{\infty\} \subset \mathrm{Int}(K(f_1))$. Then there exists an $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq m$, setting $f_i = g_i^n, i = 1, \dots, s+1$, the element $(f_1, \dots, f_{s+1})$ satisfies assumptions (1)(2)(3) and $P(\langle f_1, \dots, f_{s+1} \rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$.*

Example 2.9. Let $g_1(z) = z^2 - 1$ and let $g_i(z) = \frac{1}{10i} z^2, i = 2, \dots, s+1$. Then $(g_1, \dots, g_{s+1})$ satisfies the assumptions of Proposition 2.8. Note that $z^2 - 1$ can be replaced by any hyperbolic element $f \in \mathscr{P}$ with connected Julia set such that $J(f) \subset \{z \in \mathbb{C} : |z| < 10\}$ and $0 \in \mathrm{Int}(K(f))$.

From one element $(g_1, \dots, g_m) \in (\mathrm{Rat})^m$ which satisfies assumptions (1)(2)(3) (with $s + 1 = m$), we obtain many elements which satisfy assumptions (1)(2)(3) of our paper as follows.

Proposition 2.10. *Let $(g_1, \dots, g_m) \in (\mathrm{Rat})^m$ with $\deg(g_i) \geq 2, i = 1, \dots, m$, and suppose that $(g_1, \dots, g_m)$ satisfies assumptions (1)(2)(3) of this paper. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $f_1, \dots, f_{s+1}$ be mutually distinct elements of $\{g_{\omega_n} \circ \cdots \circ g_{\omega_1} \mid (\omega_1, \dots, \omega_n) \in \{1, \dots, m\}^n\}$ where $s \geq 1$. Then we have the following.*

- (I) *$(f_1, \dots, f_{s+1})$ satisfies assumptions (1)(2)(3) of this paper. Thus all statements in Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and Corollaries 1.6, 1.7, 1.9 and 1.11 in Section 1 hold for $(f_1, \dots, f_{s+1})$, for every minimal set L of $\langle f_1, \dots, f_{s+1} \rangle$ and for every $\mathbf{p} = (p_1, \dots, p_s) \in (0,1)^s$ with $\sum_{i=1}^s p_i < 1$.*
- (II) *If, in addition to the assumption, $(f_1, \dots, f_{s+1}) \in \mathscr{P}^{s+1}$, then statement (1) in Corollary 1.8 holds for $(f_1, \dots, f_{s+1})$ and for every $\mathbf{p}$, and statement (2) in Corollary 1.8 holds for $(f_1, \dots, f_{s+1})$ and for every $\mathbf{p}$ provided that one of (a)(b)(c) in the assumption of Corollary 1.8 (2) holds.*
- (III) *If, in addition to the assumption of our proposition, $(g_1, \dots, g_m) \in \mathscr{P}^m$ and $P(\langle g_1, \dots, g_m \rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, then $P(\langle f_1, \dots, f_{s+1} \rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. Thus, statement (2) in Corollary 1.8 holds for $(f_1, \dots, f_{s+1})$ and for every $\mathbf{p}$.*

Regarding Remark 1.15, we also have the following.

Lemma 2.11. *Let $s \geq 1$ and let $I = \{1, \dots, s+1\}$. Then the set*

$$\{(f_i)_{i \in I} \in \mathscr{P}^I : (f_i)_{i \in I} \text{ satisfies assumptions (1)(2)(3) and } P(\langle f_1, \dots, f_{s+1} \rangle) \setminus \{\infty\} \text{ is bounded in } \mathbb{C}\}$$

is open in $\mathscr{P}^I$.

We remark that the above examples, propositions and lemma in this section and Remark 1.15 imply that we have plenty of examples to which we can apply the results in Section 1.

We give examples to which we can apply Corollary 1.11.

Lemma 2.12. *Let* $(g_1,\ldots,g_{s+1})$ *be an element which satisfies assumptions (1)(2)(3). Let* $\mathbf{p}=(p_i)_{i=1}^{s}\in(0,1)^s$ *with* $\sum_{i=1}^{s+1}p_i<1$. *Let* $p_{s+1}=1-\sum_{i=1}^{s}p_i$. *Then there exists an* $m\in\mathbb{N}$ *such that for every* $n\in\mathbb{N}$ *with* $n\geq m$, *setting* $f_i=g_i^n, i=1\ldots,s+1$, *and setting* $G:=\langle f_1,\ldots,f_{s+1}\rangle$, *we have that* $(f_1,\ldots,f_{s+1})$ *satisfies assumptions (1)(2)(3) and* $p_i\min_{z\in f_i^{-1}(J(G))}\|f_i'(z)\|>1$ *for every* $i=1,\ldots,s+1$. *Thus, for every minimal set* L *of* $\langle f_1,\ldots,f_{s+1}\rangle$, *and for every* $z\in J(G)$, *we have that every non-trivial* $C\in\mathscr{C}$ *satisfies* $\mathrm{H\ddot{o}l}(C,z)\leq\alpha_+<1$ *and* C *is not differentiable at* z.

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